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A supersymmetry approach to billiards with randomly distributed scatterers: II. Correlations

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Abstract

In a previous contribution (Stöckmann H J 2002 *J. Phys. A: Math. Gen.* **35** 5165), the density of states was calculated for a billiard with randomly distributed delta-like scatterers, doubly averaged over the positions of the impurities and the billiard shape. This result is now extended to the k-point correlation function. Using supersymmetric methods, we show that the correlations in the bulk are always identical to those of the Gaussian unitary ensemble (GUE) of random matrices. In passing from the band centre to the tail states, the density of states is depleted considerably and the two-point correlation function shows a gradual change from the GUE behaviour to that found for completely uncorrelated eigenvalues. This can be viewed as similar to a mobility edge.

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1. Introduction

The theory of random matrices provides a schematic, but powerful statistical model for a wide class of spectral problems in complex systems, for reviews see [1–3]. In particular, there is overwhelming evidence for the fact that the spectral fluctuations of a quantum system whose classical counterpart is fully chaotic are described by the Gaussian ensemble of random matrices, i.e. by the Gaussian unitary ensemble (GUE) in the absence of time reversal invariance and by the Gaussian orthogonal ensemble (GOE) if time reversal invariance holds and the spectrum is free of Kramers degeneracies [4]. On the other hand, the fluctuation properties for quantum systems whose classical counterparts are regular ought to be different, and often of the Poisson type. Many systems show mixed fluctuation properties and transitions from regular to chaotic behaviour.

Quantum billiards are ideal systems for the study of spectral fluctuation properties. Billiards are said to be ballistic because the classical dynamics and the quantum spectra are exclusively determined by the shape of the boundary. Whereas such ballistic systems are

well understood, the situation is less clear for disordered systems. In particular, there are many open questions concerning the localization–delocalization transition if disorder is varied. From the one-parameter scaling hypothesis [5] it is generally accepted that in one- and two-dimensional systems all states are localized, but analytic proofs exist only for one-dimensional systems (see [6] for a review). There are a number of works using supersymmetric techniques, where the change of the wavefunction amplitude statistic is studied with the reciprocal conductance as a perturbation parameter [7], but up to now there is no closed theory covering the full range from localized to delocalized wavefunctions. On the other hand there are microwave experiments showing a clear localization–delocalization transition with frequency [8, 9].

This was the motivation of a previous publication [10], hereafter denoted I, to tackle the problem by an alternative approach. Instead of the usually applied nonlinear σ-model the more explicit system of a billiard with randomly distributed scatterers was studied. This approach generalized a model introduced by Bogomolny *et al* [11]. The average over disorder was achieved with the help of a trick using the conjecture that a typical wavefunction can be viewed as a random superposition of plane waves [12]. Thereby, no supersymmetric field variables are needed which Efetov used to construct his nonlinear σ-model [13]. It avoids as well the complications of diagrammatic expansions of Green functions and summations of ladder diagrams [6]. It was already conjectured in I that there should be a localization–delocalization transition with increasing number of scatterers. In the present work further arguments are given that for a sufficiently large number of scatterers there is indeed a mobility edge, separating the band from the tail states, where such a transition takes place. This effect is accompanied by a considerable depletion of the density of states. There is a fundamental difference to the σ-model which will be discussed.

The paper is organized as follows. In section 2 the main results of I are recapitulated and the k-point correlation function is calculated generalizing a method developed in [15]. In section 3 the results are specialized to the strong coupling limit, and it is shown that everywhere within the band random-matrix results are recovered. In section 4 the behaviour of the k-point correlation close to the band edge is studied. The two-point correlation function in particular shows a transition from GUE behaviour to that of completely uncorrelated eigenvalues suggesting that there is indeed a mobility edge.

2. The model and its supersymmetric evaluation

We set up the model in section 2.1 and map it onto the superspace in section 2.2. The kernel determining all correlation functions is calculated exactly in section 2.3. A Christoffel–Darboux formula for the kernel is worked out in section 2.4.

2.1. Setup of the model

In I the density of states was calculated for a billiard with randomly distributed scatterers, averaged over the positions of the scatterer. The system was described by the Hamiltonian

$$H = H_0 + V \tag{1}$$

where H_0 is the operator of kinetic energy, and V is the scattering potential. Assuming L point-like scatterers at positions $\vec{r}_l$, we have

$$V(\vec{r}) = 4\pi\lambda \sum_{l=0}^{L} \delta(\vec{r} - \vec{r}_l). \tag{2}$$

Using standard supersymmetric techniques, the density of states was expressed as the derivative

$$\rho(E) = \frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}J}\,\mathrm{Im}\langle Z(E+J, E-J)\rangle\Big|_{J=0} \tag{3}$$

of the generating function

$$Z(E_1, E_2) = \int \mathrm{d}[x]\exp\left(\mathrm{i}\sum_{\alpha\beta}[(E_{1+}\delta_{\alpha\beta} - (H_0)_{\alpha\beta})x_\alpha^* x_\beta + (E_{2+}\delta_{\alpha\beta} - (H_0)_{\alpha\beta})\xi_\alpha^*\xi_\beta]\right) M^L \tag{4}$$

with the volume element $\mathrm{d}[x] = \prod_{\alpha=1}^{N}\mathrm{d}x_\alpha^*\,\mathrm{d}x_\alpha\,\mathrm{d}\xi_\alpha^*\,\mathrm{d}\xi_\alpha$. Here, the quantity M is given by

$$M = \left\langle \exp\left(-4\pi\mathrm{i}\lambda\sum_{\alpha\beta}\psi_\alpha^*(r)\psi_\beta(r)(x_\alpha^* x_\beta + \xi_\alpha^*\xi_\beta)\right)\right\rangle \tag{5}$$

where the brackets denote the average over the scatterer position. To perform the average, in I a trick was applied by replacing the average over the positions by an integral over the wavefunction amplitudes ψ at the positions of the scatterers with the amplitude probability density $p(\psi)$ as a weight function. For the latter a Gaussian distribution was taken typically for chaotic billiards [12, 14]. In the next step, a second average was performed by replacing the billiard spectrum with that of a random matrix from the GUE of rank N.

As a result, a simple analytic expression was obtained for the density of states. For $L > N$ a qualitative change in the density of states was observed suggesting a localization–delocalization transition. In the following the results of I will be generalized to the calculation of the k-point correlation function, and further evidence will be presented of the existence of localized states and a certain type of mobility edge within the present model.

2.2. Supersymmetric matrix model

To compute the k-level correlation functions of k energies E_p, $p = 1, \ldots, k$, we combine and extend the procedures outlined in I and [15]. We construct the functions $\hat{R}_k(E_1, \ldots, E_k)$ obtained by averaging over the product of k Green functions, including their real parts. For example, the density defined in equation (3) is the imaginary part of $\hat{R}_1(E)$. The correlation functions $R_k(E_1, \ldots, E_k)$ for the imaginary parts only can be calculated as proper linear combinations from the functions $\hat{R}_k(E_1, \ldots, E_k)$. The latter are given as the derivatives

$$\hat{R}_k(E_1, \ldots, E_k) = \frac{1}{(2\pi)^k}\prod_{p=1}^{k}\frac{\partial}{\partial J_p}\langle Z_k(E+J)\rangle\Big|_{J=0} \tag{6}$$

of the generating function

$$\langle Z_k(E+J)\rangle = 2^{k(k-1)}\int \mathrm{d}[S]\exp\left(\frac{N}{2\pi^2}\,\mathrm{Tr}\,S^2 + \mathrm{Tr}\,S(E+J)\right)\frac{\mathrm{Det}^N S}{\mathrm{Det}^L(1_{2k} + \lambda S)} \tag{7}$$

with respect to k source variables J_p, $p = 1, \ldots, k$. Energies and source variables are ordered in diagonal matrices $E = \mathrm{diag}(E_1, E_1, \ldots, E_k, E_k)$ and $J = \mathrm{diag}(-J_1, +J_1, \ldots, -J_k, +J_k)$. The generating function (7) is the straightforward extension of the generating function used in I to arbitrary k. To keep with the notation in I, we introduced the $2k \times 2k$ Hermitian supermatrix S which can be mapped onto the supermatrix σ used in [15] by exchanging its bosonic and fermionic eigenvalues. Moreover, we use the symbols Tr and Det to indicate the supertrace and superdeterminant.

As in [15], the supersymmetric extension of the Itzykson–Zuber integral can be employed to reduce the generating function to an integral over the fermionic and bosonic eigenvalues $\mathrm{i}s_{p2}$, $p = 1, \ldots, k$ and s_{p1}, $p = 1, \ldots, k$, respectively. This is so because the term coupling S and $E + J$ is the only one in the integrand which is not invariant under a unitary transformation of S. Again, as in [15], an easy evaluation of the derivatives with respect to the source variables is possible and we arrive at

$$\hat{R}_k(E_1, \ldots, E_k) = \frac{1}{(-\pi^2)^k} \int \mathrm{d}[s] B_k(s) \exp\left(\frac{N}{2\pi^2} \operatorname{Tr} s^2 + \operatorname{Tr} sE\right) \frac{\operatorname{Det}^N s^+}{\operatorname{Det}^L (1_{2k} + \lambda s)}. \tag{8}$$

We collect the eigenvalues in the diagonal matrix $s = \operatorname{diag}(\mathrm{i}s_{12}, \ldots, \mathrm{i}s_{k2}, s_{11}, \ldots, s_{k1})$. We note that the bosonic eigenvalues carry a small imaginary increment, $s^+_{p1} = s_{p1} + \mathrm{i}\eta$, where it is necessary. It is sent to zero at an appropriate point of the calculation. Keeping this in mind, we can integrate all eigenvalues over the entire real axis. This is equivalent to the choice of the integration contour in I. The function

$$B_k(s) = \det\left[\frac{1}{s_{p1} - \mathrm{i}s_{q2}}\right]_{p,q=1,\ldots,k} \tag{9}$$

in equation (7) is the square root of the Jacobian which is due to the change of variables from the Cartesian coordinates in S to eigenvalues s and angles. It is a determinant which couples one bosonic and one fermionic eigenvalue in each of its elements. Expanding the determinant,

$$B_k(s) = \sum_{\pi} \varepsilon(\pi) \prod_{l=1}^{k} \frac{1}{s_{pl} - \mathrm{i}s_{q\pi(l)}} \tag{10}$$

where the sum is over all permutations π, and $\varepsilon(\pi) = \pm 1$ for even, and odd permutations, respectively, the integrations in equation (8) factorize into products of double integrals, each over one bosonic and one fermionic variable. The result can again be written in terms of a determinant

$$\hat{R}_k(E_1, \ldots, E_k) = \det[\hat{C}_{NL}(E_p, E_q)]_{p,q=1,\ldots,k} \tag{11}$$

with a kernel given by

$$\hat{C}_{NL}(E_p, E_q) = -\frac{1}{\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\mathrm{d}s_1\, \mathrm{d}s_2}{s_1 - \mathrm{i}s_2} \times \exp\left(-\frac{N}{2\pi^2}\left(s_1^2 + s_2^2\right) + \mathrm{i}s_2 E_q - s_1 E_p\right) \left(\frac{1 + \lambda s_1}{1 + \lambda \mathrm{i}s_2}\right)^L \left(\frac{\mathrm{i}s_2}{s_1^+}\right)^N. \tag{12}$$

We suppress the indices p and q in the integration variables. Thus, the correlation functions have a determinant structure which is an immediate consequence of the determinant (9). In full analogy with [15], we obtain the correlation functions $R_k(E_1, \ldots, E_k)$ by replacing $1\big/\left(s_1^+\right)^N$ in equation (12) with its imaginary part $\operatorname{Im} 1\big/\left(s_1^+\right)^N$. As in I, we rescale the energies and the strength parameter according to

$$\varepsilon_p = \frac{\pi}{\sqrt{2N}} E_p \qquad \text{and} \qquad \alpha = \frac{\sqrt{N/2}}{\pi\lambda}. \tag{13}$$

On this scale, the correlation functions are given by

$$R_k(\varepsilon_1, \ldots, \varepsilon_k) = \det[C_{NL}(\varepsilon_p, \varepsilon_q)]_{p,q=1,\ldots,k} \tag{14}$$

where the kernel now reads

$$C_{NL}(\varepsilon_p,\varepsilon_q) = -\frac{1}{\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\mathrm{d}s_1\,\mathrm{d}s_2}{s_1-\mathrm{i}s_2}\exp\left(-\left(s_1^2+s_2^2\right)+2\mathrm{i}s_2\varepsilon_q-2s_1\varepsilon_p\right)$$
$$\times\left(\frac{\alpha+s_1}{\alpha+\mathrm{i}s_2}\right)^L(\mathrm{i}s_2)^N\mathrm{Im}\,\frac{1}{\left(s_1^+\right)^N}. \tag{15}$$

Due to the rescaling (13), we obtain the kernel for the GUE correlation functions exactly in the form given in [15], if we consider the limit $\lambda\to 0$, i.e. $\alpha\to\infty$, or, equivalently, $L=0$. We note that the scaling factor in equation (13) is precisely the GUE mean level spacing $\pi/\sqrt{2N}$ in the centre of the semicircle.

In this derivation, we have omitted an Efetov–Wegner or a Rothstein contribution [13, 15–17] which adds to the real part of $\hat{C}_{NL}(E_p,E_q)$. The functions $C_{NL}(E_p,E_q)$, the main objects of our interest, are not affected.

2.3. Exact computation of the kernel

Extending the methods of [15], the kernel can be evaluated exactly for all values of N, L and α. We define the functions

$$u_{NL}(\varepsilon) = \frac{(-2\mathrm{i})^N\alpha^L}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\mathrm{d}s_2\exp(-(s_2-\mathrm{i}\varepsilon)^2)\frac{s_2^N}{(\alpha+\mathrm{i}s_2)^L}$$
$$v_{NL}(\varepsilon) = \frac{(-1)^{N+1}N!}{\pi\alpha^L}\exp(\varepsilon^2)\int_{-\infty}^{+\infty}\mathrm{d}s_1\exp(-(s_1+\varepsilon)^2)(\alpha+s_1)^L\mathrm{Im}\,\frac{1}{\left(s_1^+\right)^{N+1}} \tag{16}$$
$$= \frac{(-1)^N}{\alpha^L}\exp(\varepsilon^2)\,\frac{\partial^N}{\partial s_1^N}\exp(-(s_1+\varepsilon)^2)(\alpha+s_1)^L\Big|_{s_1=0}$$

which reduce to the Hermite polynomials $H_N(\varepsilon)$ for $L=0$ or, equivalently, for $\alpha\to\infty$. In the appendix, some properties of these functions are compiled. We now express $\mathrm{Im}\,1\big/\left(s_1^+\right)^N$ in equation (15) as $\partial^{N-1}\delta(s_1)/\partial s_1^{N-1}$ and integrate by parts until the $(N-1)$-fold derivative with respect to s_1 acts on all s_1 dependent terms in the integrand. After applying Leibniz' rule for multiple derivatives of products, we can insert the second form of the function $v_{NL}(\varepsilon)$ into equation (15). The s_2 integration then yields just the function $u_{NL}(\varepsilon)$ and we arrive at

$$C_{NL}(\varepsilon_p,\varepsilon_q) = \frac{1}{\sqrt{\pi}}\exp\left(-\varepsilon_q^2\right)\sum_{n=0}^{N-1}\frac{1}{2^n n!}v_{nL}(\varepsilon_p)u_{nL}(\varepsilon_q). \tag{17}$$

Thus, we have expressed the kernel and all correlations in terms of the functions $v_{NL}(\varepsilon_p)$ and $u_{NL}(\varepsilon_q)$. Formula (17) is a generalization of the corresponding expression for the GUE. We mention in passing that one also derives

$$C_{NL}(\varepsilon_p,\varepsilon_q) = \frac{(-1)^{N-1}}{2^{N-1}(N-1)!\sqrt{\pi}}\int_0^\infty\exp(-(\varepsilon_q+t)^2)u_{NL}(\varepsilon_q+t)v_{(N-1)L}(\varepsilon_p+t)\,\mathrm{d}t \tag{18}$$

which again generalizes the corresponding expression for the GUE in [15]. The result (18) involves only the orders N and $N-1$ of the functions $v_{NL}(\varepsilon_p)$ and $u_{NL}(\varepsilon_q)$, which are not even orthogonal.

2.4. A Christoffel–Darboux formula for $C_{NL}(\varepsilon_p,\varepsilon_q)$

For $L=0$ or, alternatively, $\alpha\to\infty$, the sum on the right-hand side of expression (17) can be performed with the result

$$C_{N0}(\varepsilon_p,\varepsilon_q)=\frac{1}{2^{N-1}(N-1)!\sqrt{\pi}}\exp\left(-\varepsilon_q^2\right)\frac{u_{(N-1)0}(\varepsilon_q)v_{N0}(\varepsilon_p)-u_{N0}(\varepsilon_q)v_{(N-1)0}(\varepsilon_p)}{\varepsilon_p-\varepsilon_q}. \tag{19}$$

This is the well-known Christoffel–Darboux formula for the Hermite polynomials. This expression is now generalized to arbitrary values of L. To this end we multiply both sides of equation (15) by $\varepsilon_p-\varepsilon_q$ and obtain in a sequence of elementary steps, including one integration by parts,

$$\begin{aligned}
(\varepsilon_p-\varepsilon_q)C_{NL}(\varepsilon_p,\varepsilon_q) &= -\frac{1}{\pi^2}\mathrm{Im}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\mathrm{d}s_1\,\mathrm{d}s_2}{s_1-\mathrm{i}s_2}(\varepsilon_p-\varepsilon_q)\\
&\quad\times\exp\left(-\left(s_1^2+s_2^2\right)+2\mathrm{i}s_2\varepsilon_q-2s_1\varepsilon_p\right)\left(\frac{\alpha+s_1}{\alpha+\mathrm{i}s_2}\right)^L\left(\frac{\mathrm{i}s_2}{s_1^+}\right)^N\\
&= -\frac{1}{\pi^2}\mathrm{Im}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\mathrm{d}s_1\,\mathrm{d}s_2}{s_1-\mathrm{i}s_2}\left[-\frac{1}{2}\left(\frac{\partial}{\partial s_1}+\frac{1}{\mathrm{i}}\frac{\partial}{\partial s_2}\right)\exp(2\mathrm{i}s_2\varepsilon_q-2s_1\varepsilon_p)\right]\\
&\quad\times\exp\left(-\left(s_1^2+s_2^2\right)\right)\left(\frac{\alpha+s_1}{\alpha+\mathrm{i}s_2}\right)^L\left(\frac{\mathrm{i}s_2}{s_1^+}\right)^N\\
&= -\frac{1}{\pi^2}\mathrm{Im}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\mathrm{d}s_1\,\mathrm{d}s_2}{s_1-\mathrm{i}s_2}\exp(2\mathrm{i}s_2\varepsilon_q-2s_1\varepsilon_p)\\
&\quad\times\left[\frac{1}{2}\left(\frac{\partial}{\partial s_1}+\frac{1}{\mathrm{i}}\frac{\partial}{\partial s_2}\right)\exp\left(-\left(s_1^2+s_2^2\right)\right)\left(\frac{\alpha+s_1}{\alpha+\mathrm{i}s_2}\right)^L\left(\frac{\mathrm{i}s_2}{s_1^+}\right)^N\right]\\
&= -\frac{1}{\pi^2}\mathrm{Im}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\mathrm{d}s_1\,\mathrm{d}s_2}{s_1-\mathrm{i}s_2}\\
&\quad\times\left[\frac{1}{2}\left(-2s_1+2\mathrm{i}s_2+\frac{L}{\alpha+s_1}-\frac{L}{\alpha+\mathrm{i}s_2}+\frac{N}{\mathrm{i}s_2}-\frac{N}{s_1^+}\right)\right]\\
&\quad\times\exp\left(-\left(s_1^2+s_2^2\right)+2\mathrm{i}s_2\varepsilon_q-2s_1\varepsilon_p\right)\left(\frac{\alpha+s_1}{\alpha+\mathrm{i}s_2}\right)^L\left(\frac{\mathrm{i}s_2}{s_1^+}\right)^N\\
&= -\frac{1}{\pi^2}\mathrm{Im}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{d}s_1\,\mathrm{d}s_2\frac{1}{2}\left(-2-\frac{L}{(\alpha+s_1)(\alpha+\mathrm{i}s_2)}+\frac{N}{\mathrm{i}s_2s_1^+}\right)\\
&\quad\times\exp\left(-\left(s_1^2+s_2^2\right)+2\mathrm{i}s_2\varepsilon_q-2s_1\varepsilon_p\right)\left(\frac{\alpha+s_1}{\alpha+\mathrm{i}s_2}\right)^L\left(\frac{\mathrm{i}s_2}{s_1^+}\right)^N.
\end{aligned} \tag{20}$$

The inconvenient denominator coupling the s_1 and the s_2 integrations has disappeared with the consequence that all integrals can be expressed in terms of the $u_{NL}(\varepsilon)$ and $v_{NL}(\varepsilon)$. A formula of the Christoffel–Darboux type is obtained,

$$\begin{aligned}
(\varepsilon_p-\varepsilon_q)C_{NL}(\varepsilon_p,\varepsilon_q) &= \frac{1}{2^N(N-1)!\sqrt{\pi}}\exp\left(-\varepsilon_q^2\right)\\
&\quad\times\left[-u_{NL}(\varepsilon_q)v_{(N-1)L}(\varepsilon_p)+u_{(N-1)L}(\varepsilon_q)v_{NL}(\varepsilon_p)\right.\\
&\quad\left.-\frac{L}{2\alpha^2}u_{N(L+1)}(\varepsilon_q)v_{(N-1)(L-1)}(\varepsilon_p)\right]
\end{aligned} \tag{21}$$

which is valid for all values of N, L and α. This is quite remarkable, because the functions $u_{NL}(\varepsilon)$ and $v_{NL}(\varepsilon)$ are not orthogonal polynomials. In a different context, similar

generalizations of the Christoffel–Darboux formula have been obtained in [21, 22] and by Strahov and Fyodorov [18] in the calculation of correlation functions of ratios and products of characteristic polynomials of Hermitian random matrices.

For $L = 0$ expression (19) for the Hermite polynomials is recovered. Another special case is obtained for the strong coupling limit. Here the Gauss functions in the integral (15) may be replaced by one with the consequence that the first term in the brackets on the right-hand side of equation (21) is missing. Furthermore in this limit the $u_{NL}(\varepsilon)$ and $v_{NL}(\varepsilon)$ can be expressed in terms of generalized Laguerre polynomials,

$$u_{NL}(\varepsilon) = \sqrt{\pi}\exp(\varepsilon^2)(-1)^N(2\alpha)^{(N+1)}\frac{N!}{(L-1)!}\exp(-z)z^{L-N-1}L_N^{(L-N-1)}(z) \tag{22}$$

$$v_{NL}(\varepsilon) = (-1)^N N!\alpha^{-N}L_N^{(L-N)}(z) \tag{23}$$

where $z = 2\varepsilon\alpha = E/\lambda$ as in I. Collecting the results we obtain from equation (21)

$$(\varepsilon_p - \varepsilon_q)C_{NL}(\varepsilon_p, \varepsilon_q) = \frac{N!}{(L-1)}\exp(-z_q)z_q^{L-N} \times\left[-L_{N-1}^{(L-N)}(z_q)L_N^{(L-N)}(z_p) + L_N^{(L-N)}(z_q)L_{N-1}^{(L-N)}(z_p)\right]. \tag{24}$$

Comparing equations (17) and (21) we obtain the following Christoffel–Darboux relation for the generalized Laguerre polynomials:

$$\sum_{n=0}^{N-1} n!x^{N-n-1}L_n^{(L-n-1)}(x)L_n^{(L-n)}(y) = N!\frac{L_N^{(L-N)}(x)L_{N-1}^{(L-N)}(y) - L_{N-1}^{(L-N)}(x)L_N^{(L-N)}(y)}{y-x}. \tag{25}$$

This is not the Christoffel–Darboux relation for the Laguerre polynomials found in compilation such as [19], but we cannot exclude that it is known in the mathematical literature.

3. Density of states and correlations via a saddlepoint approximation

In section 3.1, we work out the density of states in the strong coupling limit. The correlations in the bulk of the spectrum are computed for arbitrary coupling in section 3.2.

3.1. Density of states in the strong coupling limit

For strong coupling $\lambda \gg 1$ or $\alpha \ll 1$ and $L > N$, the density of states was evaluated in I by means of a WKB approximation to leading order in $L > N \gg 1$. Here we show that this is equivalent to a saddlepoint approximation. For $k = 1$, we write the generating function (17) in the form

$$\begin{aligned}\langle Z_1(\varepsilon + J)\rangle &= \int \mathrm{d}[S]\exp(\mathcal{L}(S, \varepsilon + J))\\ \mathcal{L}(S, \varepsilon + J) &= \mathrm{Tr}\,S^2 + 2\,\mathrm{Tr}\,S(\varepsilon + J) + N\,\mathrm{Tr}\ln S - L\,\mathrm{Tr}\ln(\alpha + S).\end{aligned} \tag{26}$$

We use the rescaled variables (13), drop the index 1 on the energy variable and write ε shorthand for $\varepsilon 1_2$. In the strong coupling limit S is of the order of α as can be seen by applying the substitution $S = \alpha S'$. The term $\mathrm{Tr}\,S^2$ is thus of the order of α^2 and may be dropped in the Lagrangian $\mathcal{L}(S, \varepsilon + J)$. In this approximation, the saddlepoint equation resulting from the condition $\mathrm{d}\mathcal{L} = 0$ at $J = 0$ reads

$$2\varepsilon + \frac{N}{s_0} - \frac{L}{\alpha + s_0} = 0 \tag{27}$$

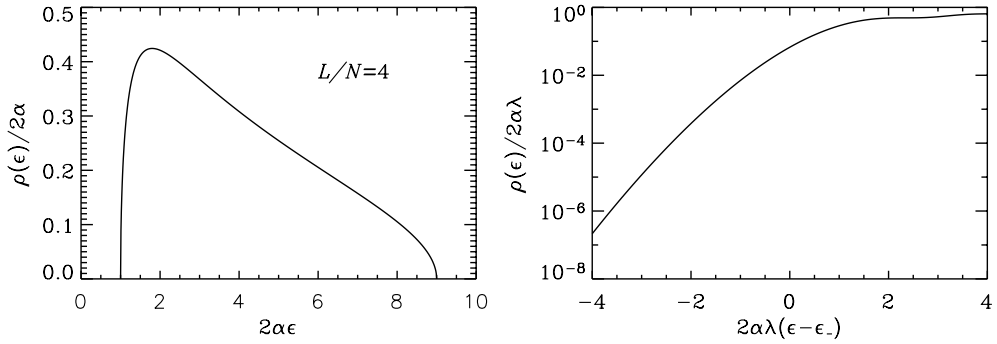

Figure 1. (*a*) Density of states in the strong coupling limit for $L/N = 4$. (*b*) Density of states in the neighbourhood of the lower band edge (for details, see section 4.2).

with s_0 standing for the two scalar saddlepoints s_{10} and is_{20}. The solutions can be written as

$$s_0 = \frac{1}{4\varepsilon}\left(-(2\varepsilon\alpha - (L-N)) \mp \mathrm{i}\sqrt{4LN - (2\varepsilon\alpha - (L-N))^2}\right). \tag{28}$$

Obviously, the imaginary part is only nonzero if the energy satisfies

$$\varepsilon_- \leqslant \varepsilon \leqslant \varepsilon_+ \qquad \text{with} \quad \varepsilon_\mp = \frac{1}{2\alpha}\left(L + N \mp 2\sqrt{LN}\right). \tag{29}$$

We now expand the Lagrangian $\mathcal{L}(S, \varepsilon + J)$ around the saddlepoints up to second order and integrate out the massive modes in a Gaussian fashion. One can convince oneself in a straightforward, but tedious, calculation that these Gaussian integrals converge as long as the condition (29) holds. At the saddlepoints, the Lagrangian is simply $4s_0J_1$ and we find from equation (6)

$$\begin{aligned}\hat{R}_1(\varepsilon) &= \frac{2}{\pi}s_0 \\ &= \frac{1}{2\pi\varepsilon}\left(-(2\varepsilon\alpha - (L-N)) + \mathrm{i}\sqrt{4LN - (2\varepsilon\alpha - (L-N))^2}\right).\end{aligned} \tag{30}$$

This is the full one-point function in the strong coupling limit. The imaginary part is the density of states which is nonzero for $\varepsilon_- \leqslant \varepsilon \leqslant \varepsilon_+$. As expected, it coincides with the WKB approximation of I. The saddlepoint approximation yields, in addition, also the real part of the one-point function. As an illustration figure 1(*a*) shows the density of states calculated from the imaginary part of $\hat{R}_1(\varepsilon)$ for $L/N = 4$.

3.2. Correlations in the bulk of the spectrum

We take advantage of a remarkable identity which connects the kernel and the generating function for $k = 1$,

$$\begin{aligned}\hat{C}_{NL}(\varepsilon_p, \varepsilon_q) &= -\frac{1}{\pi(\varepsilon_p - \varepsilon_q)}\langle Z_1(\mathcal{E})\rangle \\ &= -\frac{1}{\pi(\varepsilon_p - \varepsilon_q)}\int \mathrm{d}[S]\exp(\mathrm{Tr}\,S^2 + \mathrm{Tr}\,S\mathcal{E})\frac{\mathrm{Det}^{\,N}S}{\mathrm{Det}^{\,L}(\alpha 1_2 + S)}\end{aligned} \tag{31}$$

with $\mathcal{E} = \mathrm{diag}(\varepsilon_p, \varepsilon_q)$. This identity which is easily derived with the supersymmetric extension of the Itzykson–Zuber integral for $k = 1$ allows us to work out the correlations on the unfolded scale by a saddlepoint approximation involving 2×2 supermatrices, i.e. in a simple Cartesian

space. A similar procedure was employed in [20] in the context of chiral random-matrix ensembles.

To begin with, we discuss the strong coupling limit of the previous section and turn to the general case later. We write

$$\mathcal{E} = \varepsilon_{pq} 1_2 + D\frac{r_{pq}}{2}\Lambda \qquad \text{with} \quad \varepsilon_{pq} = \frac{\varepsilon_p + \varepsilon_q}{2} \quad \text{and} \quad Dr_{pq} = \varepsilon_p - \varepsilon_q. \tag{32}$$

Here, we introduced the metric $\Lambda = \mathrm{diag}(+1, -1)$ and, anticipating the steps to come, the local mean level spacing $D = 1/R_1(\varepsilon_{pq})$ which defines the unfolded scale. We use the form (26) for the generating function with $\varepsilon + J$ replaced by $\mathcal{E}$ to evaluate equation (31). In the strong coupling limit, we neglect the term $\mathrm{Tr}\, S^2$. Although $\alpha \ll 1$ in this limit, we do not make any assumption about its value in the present discussion. The saddlepoints are the stable points of the integrand in an asymptotic $1/N$ expansion. The unfolded correlations live on the local scale of the mean level spacing D. Thus, we have to keep $r_{pq} = (\varepsilon_p - \varepsilon_q)/D$ fixed in the asymptotic expansion for the calculation of the unfolded correlations. The energy difference $\varepsilon_p - \varepsilon_q$ itself appears in the integrand. As it is given by Dr_{pq}, and as the mean level spacing D vanishes in the limit $N \to \infty$, the energy difference cannot yield a contribution to the saddlepoints and we may neglect it when calculating them. Thus, we are left with exactly the same problem as in the previous section, only ε is replaced by ε_{pq}. This implies that the integrals over the massive modes converge in the nonzero region of the spectrum and, moreover, that the only non-vanishing contribution to the correlations comes from the term $\mathrm{Tr}\, S\mathcal{E}$ in the Lagrangian. Collecting everything, we find

$$\hat{c}_{NL}(r_{pq}, \tilde{r}_{pq}) = \lim_{L>N\to\infty} D\hat{C}_{NL}(\varepsilon_p, \varepsilon_q) = \exp(\pi\tilde{r}_{pq})\frac{\exp(\mathrm{i}\pi r_{pq})}{\pi r_{pq}} \tag{33}$$

for the kernel on the unfolded scale. We note that the result depends on $\tilde{r}_{pq} = (\varepsilon_p - \varepsilon_q)\widetilde{R_1}(\varepsilon_{pq})$ where $\widetilde{R_1}(\varepsilon_{pq}) = \mathrm{Re}\,\widehat{R_1}(\varepsilon_{pq})$ is the real part of the one-point function. As discussed in [21], an Efetov–Wegner or a Rothstein term has to be added to equation (33). It affects only the real part and reads $-1/\pi r_{pq}$. For the correlation functions involving the imaginary parts of the Green functions, we only need the imaginary part

$$c_{NL}(r_{pq}, \tilde{r}_{pq}) = \mathrm{Im}\,\hat{c}_{NL}(r_{pq}, \tilde{r}_{pq}) = \exp(\pi\tilde{r}_{pq})\frac{\sin\pi r_{pq}}{\pi r_{pq}} \tag{34}$$

which consists of the GUE sine kernel and an exponential function depending on $\tilde{r}_{pq}$. Both variables, $\tilde{r}_{pq}$ and r_{pq}, are odd under the exchange of the indices p and q. Thus, the sine kernel stays unchanged, while the exponential function acquires a sign in its argument. This implies for the correlation function on the local scale

$$\begin{aligned} X_k(r_{12}, r_{13}, \ldots, r_{(k-1)k}) &= \lim_{L>N\to\infty} D^k R_k(\varepsilon_1, \ldots, \varepsilon_k) \\ &= \det\left[\frac{\sin\pi r_{pq}}{\pi r_{pq}}\right]_{p,q=1,\ldots,k} \end{aligned} \tag{35}$$

which is identical to the standard GUE correlations.

The previous derivation is for the strong coupling limit. In the following, we present a general discussion of the correlations. We write the kernel as the convolution

$$\begin{aligned} \hat{C}_{NL}(\varepsilon_p, \varepsilon_q) &= \frac{1}{\pi}\int_{-\infty}^{+\infty} \mathrm{d}y_p \exp\left(-y_p^2\right) \int_{-\infty}^{+\infty} \mathrm{d}y_q \exp\left(-y_q^2\right) \hat{B}_{NL}(\varepsilon_p + y_p, \varepsilon_q + \mathrm{i}y_q) \\ \hat{B}_{NL}(z_p, z_q) &= -\frac{1}{\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\mathrm{d}s_1\,\mathrm{d}s_2}{s_1 - \mathrm{i}s_2} \exp\left(-\left(s_1^2 + s_2^2\right) + \mathrm{i}\,s_2 z_q - s_1 z_p\right) \\ &\quad \times \left(\frac{\alpha + s_1}{\alpha + \mathrm{i}s_2}\right)^L \left(\frac{\mathrm{i}s_2}{s_1^+}\right)^N. \end{aligned} \tag{36}$$

Obviously, the kernel $\hat{B}_{NL}(z_p, z_q)$ is the kernel of the strong coupling limit. However, it emerges due to the convolution. Thus, we do not need to assume that $L > N$. In the following, we only assume that both numbers, L and N, are large. Moreover, we make no assumption about α. As we have seen in the previous discussion, this kernel, here denoted by $\hat{B}_{NL}(z_p, z_q)$, leads to standard GUE correlations on the unfolded scale. One might argue that this does not necessarily carry over to the present case, because the arguments $z_p = \varepsilon_p + y_p$ and $z_q = \varepsilon_q + \mathrm{i}y_q$ contain the integration variables y_p and y_q. However, as we are only interested in the fluctuations, we only need to consider the integration variables on this scale. Thus, we may neglect them for the determination of the saddlepoints. After assembling things properly, we arrive at

$$\begin{aligned}\hat{c}_{NL}(r_{pq}, \tilde{r}_{pq}) &= \lim_{N\to\infty} D\hat{C}_{NL}(\varepsilon_p, \varepsilon_q) \\ &= \frac{1}{\pi}\int_{-\infty}^{+\infty} \mathrm{d}y_p \exp\left(-y_p^2\right) \int_{-\infty}^{+\infty} \mathrm{d}y_q \exp\left(-y_q^2\right) \\ &\quad \times \exp(\pi(\tilde{r}_{pq} + (y_p - \mathrm{i}y_q)\widetilde{R}_1(\varepsilon_{pq}))) \frac{\exp(\mathrm{i}\pi(r_{pq} + (y_p - \mathrm{i}y_q)/D))}{\pi(r_{pq} + (y_p - \mathrm{i}y_q)/D)}. \end{aligned} \tag{37}$$

As we are only interested in the imaginary part, we may again ignore the Efetov–Wegner or Rothstein term. The imaginary part can be obtained from the difference of a retarded and an advanced Green function. The two Green functions yield the same kernels, apart from a sign change in the argument of the exponential function in the numerator, $\exp(\pm\mathrm{i}\pi(r_{pq} + (y_p - \mathrm{i}y_q)/D))$. Hence, only the difference of these two exponential functions, the sine function, enters. This is equivalent to taking the imaginary part of equation (37) while formally ignoring the imaginary unit coming with the variable y_q. Thus, we find

$$\begin{aligned} c_{NL}(r_{pq}, \tilde{r}_{pq}) &= \mathrm{Im}\, \hat{c}_{NL}(r_{pq}, \tilde{r}_{pq}) \\ &= \frac{1}{\pi}\int_{-\infty}^{+\infty} \mathrm{d}y_p \exp\left(-y_p^2\right) \int_{-\infty}^{+\infty} \mathrm{d}y_q \exp\left(-y_q^2\right) \\ &\quad \times \exp(\pi(\tilde{r}_{pq} + (y_p - \mathrm{i}y_q)\widetilde{R}_1(\varepsilon_{pq}))) \frac{\sin(\pi(r_{pq} + (y_p - \mathrm{i}y_q)/D))}{\pi(r_{pq} + (y_p - \mathrm{i}y_q)/D)} \\ &= \exp(\pi\tilde{r}_{pq}) \frac{\sin \pi r_{pq}}{\pi r_{pq}} \end{aligned} \tag{38}$$

where the integrals over y_p and y_q were calculated as in [22]. Hence, the correlations are, once more, of the standard GUE type. Some comments are in order. First, it should be clear that the mean level spacing D in the above calculation was formally the one of the strong coupling limit and has thus to be smoothly adjusted when going into another regime. Therefore, our line of argument is correct only if we are always in the bulk of the spectrum, i.e. far away from any possible edges or gaps. Second, the discussion beyond the strong coupling limit could also be done in a saddlepoint approximation of the full expression (31). This, however, leads to a most inconvenient third-order saddlepoint equation. In the approach chosen here we avoid this and also gain the insight that the strong coupling limit and the general case are related via a convolution. Third, we emphasize that the connection (31) between the kernel and the generating function for $k = 1$ simplifies the calculations enormously: the saddlepoints are isolated, no Goldstone modes occur. Furthermore, all correlations are treated at once.

4. The band edges

From the pioneering work of Mott and Anderson it is known that in disordered systems there are no sharp band edges for the density of states. There is a mobility edge instead separating the delocalized states in the band from the localized ones in the tails. The mathematical origin of the band edges is due to the fact that depending on some parameter the two solutions of the saddle point equation (27) change from complex conjugate to real. This behaviour is generic, though in the present model the band edges are only an artefact of the finite rank of the matrices. It is therefore worthwhile to study the regime of the band edges somewhat more in detail.

After obtaining a WKB approximation for the kernel in section 4.1, we work out density of states and correlations in section 4.2.

4.1. A WKB approximation for $C_{NL}(\varepsilon_p, \varepsilon_q)$

To keep the discussion simple, we again concentrate on the strong coupling limit. The starting point is the Christoffel–Darboux relation (24) for $C_{NL}(\varepsilon_p, \varepsilon_q)$ holding in this limit. Using standard relations for the generalized Laguerre polynomials, it can be written in the alternative form

$$(\varepsilon_p - \varepsilon_q)C_{NL}(\varepsilon_p, \varepsilon_q) = 2\alpha \frac{N!}{(L-1)} \exp(-z_q) z_q^{L-N} \times \left[L_N^{(L-N-1)'}(z_q) L_N^{(L-N-1)}(z_p) - L_N^{(L-N-1)}(z_q) L_N^{(L-N-1)'}(z_p) \right] \tag{39}$$

where $z_{p/q} = 2\alpha\varepsilon_{p/q}$, which is somewhat more suitable for the present purpose. Following I, we write $L_N^{(L-N-1)}(z)$ as

$$L_N^{(L-N-1)}(z) = \sqrt{\frac{(L-1)!}{N!}} \exp(z/2) z^{-\frac{L-N}{2}} f(z) \tag{40}$$

where $f(z)$ is a solution of

$$\begin{aligned} &f''(z) + q^2(z) f(z) = 0 \\ &q^2(z) = \frac{N+L}{2z} - \frac{1}{4} + \frac{1-(L-N-1)^2}{4z^2}. \end{aligned} \tag{41}$$

$q^2(z)$ may be written as

$$q^2(z) = -\frac{1}{4z^2}(z - z_-)(z - z_+) \tag{42}$$

where

$$z_\pm = N + L \pm \sqrt{4NL + 2(L-N)} \approx N + L \pm 2\sqrt{NL}. \tag{43}$$

We note that this is the same expression, which was obtained above from the saddlepoint approximation for the band edges in the strong coupling limit (see section 3.1).

For $z_- \ll z \ll z_+$ the WKB solution of equation (41) is given by

$$f(z) = \sqrt{\frac{1}{\pi q(z)}} \cos\left[Q(z) - \frac{\pi}{4} \right] \tag{44}$$

where

$$Q(z) = \int_{z_-}^{z} q(t)\, \mathrm{d}t. \tag{45}$$

(To be concise we restrict the discussion to the neighbourhood of the lower edge, but it is straightforward to transfer all results to the upper edge as well.) Inserting this into equation (39), we recover the result for $C_{NL}(\varepsilon_p, \varepsilon_q)$ obtained in section 3.2 by means of the saddlepoint technique. For $z \ll z_-$ the corresponding expression reads

$$f(z) = \frac{1}{2}\sqrt{\frac{1}{\pi|q(z)|}}\exp[-|Q(z)|]. \tag{46}$$

Inserting expression (46) into (39) one notes that $C_{NL}(\varepsilon_p, \varepsilon_q)$ vanishes within the limits of the WKB approximation applied. To describe this regime appropriately, one would have to go to the next WKB order.

We do not proceed further in this direction, but concentrate on the immediate neighbourhood of the lower edge which is not covered by equations (44) and (46). Linearizing $q(z)$ close to z_-,

$$q(z) = \sqrt{\frac{z_+ - z_-}{4z_-^2}(z - z_-)} \tag{47}$$

equation (41) can be solved with the result

$$\begin{aligned} f(z) &= \frac{1}{\sqrt{\lambda}}\mathrm{Ai}[\lambda(z_- z)] \\ \lambda &= \left(\frac{\sqrt{z_+ - z_-}}{2z_-}\right)^{2/3} = \left(\frac{(NL)^{1/4}}{N + L - 2\sqrt{NL}}\right)^{2/3} \end{aligned} \tag{48}$$

where $\mathrm{Ai}(z)$ is the Airy function. With the factor $\lambda^{-1/2}$ the asymptotic behaviour of the Laguerre polynomials is reproduced correctly by equation (40). This can be shown by techniques described, e.g. in chapter 9.3 of [23]. Collecting the results, we obtain from equation (39)

$$\begin{aligned} C_{NL}(\varepsilon_p, \varepsilon_q) = 2\alpha \exp\left(-\frac{z_q - z_p}{2}\right)\left(\frac{z_q}{z_p}\right)^{(L-N)/2} \\ \times\left[\frac{f'(z_q)f(z_p) - f(z_q)f'(z_p)}{z_p - z_q} - (L - N)\frac{f(z_p)f(z_q)}{2z_p z_q}\right]. \end{aligned} \tag{49}$$

The second term on the right-hand side vanishes for $L \to \infty$, since z_p, z_q are of order $\mathrm{O}(N + L)$, and will be discarded in the following. Essentially the same approach to describe the behaviour of correlation functions close to the band edges was applied by Akemann and Fyodorov [24] in the study of characteristic polynomials.

4.2. The density of states and the k-point correlation function

The density of states is obtained from equation (49),

$$\rho(\varepsilon) = C_{NL}(\varepsilon, \varepsilon) = 2\alpha\{[f'(2\alpha\varepsilon)]^2 - f(2\alpha\varepsilon)f''(2\alpha\varepsilon)\}. \tag{50}$$

For the regime close to the lower band edge we obtain by inserting expression (48) for $f(z)$,

$$\rho(\varepsilon) = 2\alpha\lambda\{[\mathrm{Ai}'(-s)]^2 + s[\mathrm{Ai}(-s)]^2\} \qquad s = 2\alpha\lambda(\varepsilon - \varepsilon_-). \tag{51}$$

Figure 1(*b*) shows a plot of the density of states in the transition regime as obtained from equation (51).

From equation (14) the two-point correlation function results as

$$R_k(\varepsilon_1, \ldots, \varepsilon_k) = (2\alpha)^k \det[c_{LN}(\varepsilon_p, \varepsilon_q)]_{p,q=1,\ldots,k} \tag{52}$$

where

$$c_{LN}(\varepsilon_p, \varepsilon_q) = \frac{f'(z_q)f(z_p) - f(z_q)f'(z_p)}{z_p - z_q}. \tag{53}$$

(The first factor on the right hand-side of equation (49) cancels in taking the determinant as is easily seen.) Just as in section 3.2, we now introduce rescaled variables

$$\varepsilon_{pq} = \frac{\varepsilon_p + \varepsilon_q}{2} \qquad Dr_{pq} = \varepsilon_p - \varepsilon_q \tag{54}$$

and a rescaled correlation function $X_k(r_{12}, r_{13}, \ldots, r_{(k-1)k}) = D^k R_k(\varepsilon_1, \ldots, \varepsilon_k)$, where $D^{-1} = \rho(\varepsilon)$ is the local density of states as given by equation (50). Note that in contrast to section 3.2 we do *not* perform the limit $L, N \to \infty$, since we are interested in particular in the behaviour close to the band edges. We then have

$$X_k(r_{12}, r_{13}, \ldots, r_{(k-1)k}) = \det[\hat{c}_{LN}(\varepsilon_{pq}, r_{pq})]_{p,q=1,\ldots,k} \tag{55}$$

with

$$\hat{c}_{LN}(\varepsilon, r) = c_{LN}\left(\varepsilon + \frac{Dr}{2}, \varepsilon - \frac{Dr}{2}\right). \tag{56}$$

Inserting for $f(z)$ expression (48), we obtain for the regime close to the lower edge

$$\hat{c}_{LN}(\varepsilon, r) = \frac{\mathrm{Ai}'(-s_+)\mathrm{Ai}(-s_-) - \mathrm{Ai}(-s_+)\mathrm{Ai}'(-s_-)}{r} \tag{57}$$

where $s_\pm = s \pm \alpha\lambda Dr$. In the limit $r \to 0$ we obtain $\hat{c}_{LN}(\varepsilon, 0) = 1$ as should be. This is a direct consequence of the differential equation $\mathrm{Ai}''(z) - z\mathrm{Ai}(z) = 0$ of the Airy function, and equation (51). For $r \to \infty$ $\hat{c}_{LN}(\varepsilon, r)$ decays according to

$$\hat{c}_{LN}(\varepsilon, r) \sim r^{-1/4} \exp\left[-\tfrac{2}{3}(\alpha\lambda Dr)^{2/3}\right] \tag{58}$$

which follows from the asymptotic behaviour of the Airy function (see also equation (46)). The transition between the two regimes is observed at $r = s/\alpha\lambda D$. With decreasing density of states $\rho(\varepsilon) = 1/D$ the transition point thus approaches $r = 0$, i.e. the eigenvalues become more and more uncorrelated. This is illustrated in figure 2 where the two-point correlation function $R_2(r) = 1 - [\hat{c}_{LN}(\varepsilon, r)]^2$ is shown for different values of $s = 2\alpha\lambda(\varepsilon - \varepsilon_-)$ in the neighbourhood of the lower band edge. In addition the GUE result is shown for comparison. We observe with decreasing s a gradual transition from a GUE behaviour to that expected for completely uncorrelated eigenvalues. This is exactly what is expected for a mobility edge: within the band the eigenvalues experience a quadratic level repulsion typically for the GUE, whereas in the tails the localization of the wavefunction leads to a suppression of the level repulsion (see, e.g., [25]).

5. Summary and conclusion

In I the density of states for the billiard with randomly distributed scatterers was calculated, doubly averaged over the disorder and shape of the billiard. We mention in passing that the resulting model shows some formal similarities to chiral random-matrix models [26]. This is due to the way how the average over the disorder is done. In the present work, the results of I are extended. We calculate the k-point correlation functions exactly. The model of I generalizes that of Bogomolny *et al* [11]. These authors considered a single scatterer in a chaotic billiard and showed that the fluctuations are chaotic. In the present contribution, we extend this study to arbitrarily many scatterers and also develop a completely different technique to derive the correlations. Generalizing the approach of [15], the correlation

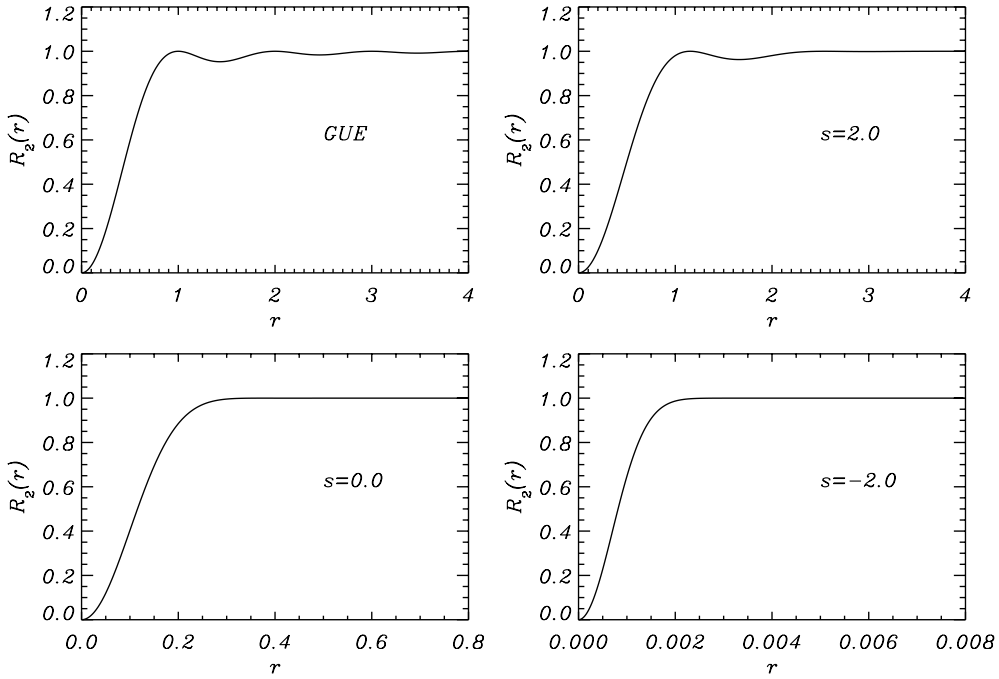

Figure 2. Two-point correlation function $R_2(r)$ for different values of $s = 2\alpha\lambda(\varepsilon - \varepsilon_-)$. For comparison the GUE result is shown as well.

functions are expressed in terms of a determinant. This determinant structure of the correlation functions is immediately obvious in the supersymmetric formulation of the model due to the form of the Berezinian in eigenvalue angle coordinates. Moreover, an explicit Christoffel–Darboux formula is given for the kernel entering the determinant.

By means of a saddlepoint approximation, we rederive the density of states in the strong coupling limit and also find the real part of the one-point function. We show that the correlation functions in the bulk of the spectrum on the scale of the local mean level spacing are, for all couplings, of the GUE type.

Applying a WKB approximation to the kernel, the correlation functions are studied close to the band edges in the strong coupling limit, where the number of scatterers is large and the scattering potential is strong. The above-mentioned saddlepoint approximation is not valid in this regime. Within the band the two-point correlation function shows a GUE behaviour, but approaching the band edges and proceeding towards the band tails the eigenvalues become more and more uncorrelated. This is exactly the fingerprint expected for a mobility edge and a localization–delocalization transition. We note that a drastic depletion of the density of states accompanies this transition. Thus, the localization–delocalization transition found in the nonlinear σ-model [13] is of a different nature. In the latter, the average is over an ensemble of white-noise correlated impurities, while two averages are performed in the present model, one over the wavefunctions at the positions of the scatterers and another one over the billiard spectrum. The resulting models are therefore different. There is a kinetic term in the nonlinear σ-model and a diffusion constant in front of it. No analogy to this is present in the model discussed here, because the average over the billiard spectrum takes care of the kinetic term.

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Appendix. Properties of the functions generalizing the Hermite polynomials

It is useful to define the functions

$$\begin{aligned}\varphi_{NL}(\varepsilon) &= \frac{\exp(-\varepsilon^2/2)}{\sqrt{2^N N!\sqrt{\pi}}} u_{NL}(\varepsilon) \\ \psi_{NL}(\varepsilon) &= \frac{\exp(-\varepsilon^2/2)}{\sqrt{2^N N!\sqrt{\pi}}} v_{NL}(\varepsilon)\end{aligned} \tag{A.1}$$

which reduce to the oscillator wavefunctions for $L = 0$ or, equivalently, for $\alpha \to \infty$. We also introduce the operators

$$A^+ = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} - \varepsilon \qquad \text{and} \qquad A^- = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} + \varepsilon \tag{A.2}$$

which act on the functions (A.1) according to

$$\begin{aligned}A^+\varphi_{NL}(\varepsilon) &= -\sqrt{2(N+1)}\varphi_{(N+1)L}(\varepsilon) \\ A^-\varphi_{NL}(\varepsilon) &= +\sqrt{2N}\varphi_{(N-1)L}(\varepsilon) + \frac{L}{\alpha}\varphi_{N(L+1)}(\varepsilon) \\ A^+\psi_{NL}(\varepsilon) &= -\sqrt{2(N+1)}\psi_{(N+1)L}(\varepsilon) - \frac{L}{\alpha}\psi_{N(L-1)}(\varepsilon) \\ A^-\psi_{NL}(\varepsilon) &= +\sqrt{2N}\psi_{(N-1)L}(\varepsilon).\end{aligned} \tag{A.3}$$

These results extend the formulae for the oscillator wavefunctions by terms involving a change of the index L. We evaluate the action of the iterated operators A^-A^+ and A^+A^- using equations (A.3), properly combine terms and arrive at the second-order differential equations

$$\begin{aligned}\left(\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2} - \varepsilon^2 + (2N+1)\right)\varphi_{NL}(\varepsilon) &= -\frac{L}{\alpha}\sqrt{2(N+1)}\varphi_{(N+1)(L+1)}(\varepsilon) \\ \left(\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2} - \varepsilon^2 + (2N+1)\right)\psi_{NL}(\varepsilon) &= -\frac{L}{\alpha}\sqrt{2N}\psi_{(N-1)(L-1)}(\varepsilon).\end{aligned} \tag{A.4}$$

These are not eigenvalue equations, because the functions on the left- and the right-hand sides have different indices. However, one can cast them into diffusion-type equations by introducing the fictitious time

$$\tau = -\ln\alpha \qquad \text{such that} \quad \alpha = \exp(-\tau). \tag{A.5}$$

A straightforward calculation yields the equations

$$\begin{aligned}\left(\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2} - \varepsilon^2 + (2N+1)\right)\varphi_{NL}(\varepsilon) &= -2\frac{\partial}{\partial\tau}\varphi_{NL}(\varepsilon) \\ \left(\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2} - \varepsilon^2 + (2N+1)\right)\psi_{NL}(\varepsilon) &= +2\frac{\partial}{\partial\tau}\psi_{NL}(\varepsilon)\end{aligned} \tag{A.6}$$

which involve the same indices on both sides.

References

[1] Mehta M L 1990 *Random Matrices* 2nd edn (San Diego, CA: Academic)
[2] Haake F 2001 *Quantum Signatures of Chaos* 2nd edn (Berlin: Springer)
[3] Guhr T, Müller-Groeling A and Weidenmüller H A 1998 *Phys. Rep.* **299** 189
[4] Bohigas O, Gianonni M J and Schmit C 1984 *Phys. Rev. Lett.* **52** 1
[5] Abrahams E, Anderson P, Licciardello D and Ramakrishnan T 1979 *Phys. Rev. Lett.* **42** 673
[6] Kramer B and MacKinnon A 1993 *Rep. Prog. Phys.* **56** 1469
[7] Mirlin A 2000 *Phys. Rep.* **326** 259
[8] Kudrolli A, Kidambi V and Sridhar S 1995 *Phys. Rev. Lett.* **75** 822
[9] Stöckmann H J *et al* 2001 *Physica* E **9** 571
[10] Stöckmann H J 2002 *J. Phys. A: Math. Gen.* **35** 5165
[11] Bogomolny E, Leboeuf P and Schmit C 2000 *Phys. Rev. Lett.* **85** 2486
[12] Berry M 1977 *J. Phys. A: Math. Gen.* **10** 2083
[13] Efetov K 1983 *Adv. Phys.* **32** 53
[14] McDonald S W and Kaufman A N 1988 *Phys. Rev.* A **37** 3067
[15] Guhr T 1991 *J. Math. Phys.* **32** 336
[16] Verbaarschot J J M, Weidenmüller H A and Zirnbauer M R 1985 *Phys. Rep.* **129** 367
[17] Rothstein M J 1987 *Trans. Am. Math. Soc.* **299** 387
[18] Strahov E and Fyodorov Y V 2002 *Preprint* math-ph/0210010
[19] Abramowitz M and Stegun I A 1970 *Handbook of Mathematical Functions* (New York: Dover)
[20] Guhr T and Wettig T 1997 *Nucl. Phys.* B **506** 589
[21] Guhr T 1993 *Nucl. Phys.* A **560** 223
[22] Guhr T 1996 *Ann. Phys.* **250** 145
[23] Morse P M and Feshbach H 1953 *Methods of Theoretical Physics* vol II (New York: McGraw-Hill)
[24] Akemann G and Fyodorov Y V 2003 *Preprint* hep-th/030495
[25] Izrailev F 1990 *Phys. Rep.* **196** 299
[26] Shuryak E V and Verbaarschot J J M 1993 *Nucl. Phys.* A **560** 306